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Construction of Association Schemes and Designs from Finite Groups*

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A necessary and sufficient condition is given for the partition of a finite group to define an association scheme. Using this condition, several classes of association schemes and balanced incomplete designs have been constructed.

1. INTRODUCTION

An m-class association scheme with v objects is defined by the following conditions:

(i) Any two objects are either first, second,..., or m-th associates.
(ii) Each object has n_i i-th associates, $i = 1, 2,..., m$.
(iii) For any pair of objects which are i-th associates, the number of p^i_{jk} objects which are j-th associates of the first and the k-th associates of the second is independent of the pair of i-th associates with which we start. We also define every object as its 0-th associate. Putting $n_i = p^0_{ii}$, $i = 0,1,..., m$, it is wellknown that the parameters, p^i_{jk}, satisfy the following conditions:

(a) $\sum_{i=1}^{m} n_i = v$,

(b) $p^0_{jk} = n_j\delta_{jk}$ and $p^i_{j0} = \delta_{ij}$, $i, j, k = 0, 1,..., m$,

(c) $p^i_{jk} = p^i_{kj}$, $i, j, k = 0, 1,..., m$,

(*d*) $p^i_{jk}\, n_i = p^j_{ik}\, n_j = p^k_{ij}\, n_k$, $i, j, k = 0, 1,..., m$,

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(e) $\sum_{j=0}^{m} p_{jk}^{i} = n_k$, independently of i, $i, k = 0, 1, \ldots, m$.

Here $\delta_{jk} = 1$ if $j = k$, 0 otherwise.

The association matrices B_0, $B_1, \ldots, B_m$ of an m-class association scheme are $v \times v$ matrices defined by

$$B_0 = I, \quad B_i = ((b_{\alpha\beta}^{i})), \quad i = 1, 2, \ldots, m, \tag{1.2}$$

where $b_{\alpha\beta}^{i} = 1$ if objects α and β are i-th associates
$= 0$ otherwise.

Clearly, B_i is a symmetric matrix with all row and column sums equal to n_i.

Lemma 2.1 of [5] states:

LEMMA 1.1. *A necessary and sufficient condition for a set of* $m + 1$ *matrices,* $\{A_0 = I_v, A_1, \ldots, A_m\}$, *of order* v *to be the set of association matrices of an* m*-class association scheme with parameters*

$$v, p_{jk}^{i}, \qquad i, j, k = 0, 1, \ldots, m, \tag{1.3}$$

is given by a set of conditions:

$$\begin{aligned} &\text{(i)} \quad \textit{each } A_i \textit{ is a symmetric } (0, 1)\textit{-matrix}, \qquad i = 0, 1, \ldots, m, \\ &\text{(ii)} \quad \sum_{i=0}^{m} A_i = J_v \ (\textit{the } v \times v \textit{ matrix of } 1\text{'s}), \qquad \textit{and} \\ &\text{(iii)} \quad A_j A_k = \sum_{i=0}^{m} p_{jk}^{i} A_i, \qquad j, k = 0, 1, \ldots, m. \end{aligned} \tag{1.4}$$

Let us divide a given set of v distinct permutation matrices of order v,

$$\Pi = \{P_0 = I_v, P_1, \ldots, P_{v-1}\}, \tag{1.5}$$

into $m + 1$ non-empty subsets

$$\Pi_0 = \{P_0\}, \Pi_1, \ldots, \Pi_m, \tag{1.6}$$

and let us put

$$A_i = \sum_{\Pi_i} P_\alpha, \qquad i = 0, 1, \ldots, m, \tag{1.7}$$

where the summation is taken over all P_α belonging to Π_i. Then, $m + 1$ matrices $A_0 = I_v$, $A_1, \ldots, A_m$ are all $v \times v$ matrices.

The conditions (i) to (iii) of (1.4), then can be rewritten as

$$
(1.8)\qquad
\begin{aligned}
&\text{(i)}\quad \sum_{\Pi_i} P_\alpha = \sum_{\Pi_i} P_\alpha^{-1}, \qquad i = 0, 1, \ldots, m,\\
&\text{(ii)}\quad \sum_{\Pi} P_\alpha = J_v, \qquad \text{and}\\
&\text{(iii)}\quad \sum_{\Pi_j} P_\beta \sum_{\Pi_k} P_\gamma = \sum_{i=0}^{m} p_{jk}^i \sum_{\Pi_i} P_\alpha, \qquad j, k = 0, 1, \ldots, m,
\end{aligned}
$$

where the summation of the right-hand side of (i) is taken over all P_α belonging to Π_i.

(1.8) are a set of necessary and sufficient conditions for the set Π of v distinct permutation matrices of (1.5) to provide an m-class association scheme with parameters (1.3).

Chakravarti and Blackwelder [1] gave a set of sufficient conditions for a set of permutation matrices Π to define an m-class association scheme. It was pointed out by Ikeda, in the discussion of the paper [1], that the condition was also necessary if Π was a group. Theorem 3.1 in [1], then, as modified by Ikeda can be stated as:

THEOREM 1.1. *Suppose that Π of* (1.5) *forms a group of order v. Let Π_i^{-1} denote $\{P_\alpha^{-1} \mid P_\alpha \in \Pi_i\}$, $i = 0, 1, 2, \ldots, m$. Then the set of conditions* (1.8) *is equivalent to the following set of conditions:*

(1.9)
(i) $\Pi_i = \Pi_i^{-1}$, $i = 0, 1, \ldots, m$,

(ii) $\sum_{\Pi} P_\alpha = J_v$, *and*

(iii) *for any given P_α belonging to Π_i, the equation*

$$P_\beta P_\gamma = P_\alpha$$

has exactly p_{jk}^i solutions (P_β, P_γ) such that $P_\beta \in \Pi_j$ and $P_\gamma \in \Pi_k$, for any j and k, $j, k = 0, 1, \ldots, m$.

Proof. Conditions (1.9) obviously imply those of (1.8). Note that

$$
(1.10)\qquad
\begin{gathered}
\sum_{\alpha=0}^{v-1} P_\alpha = J_v \qquad \text{and} \qquad \sum_{\alpha=0}^{v-1} c_\alpha P_\alpha = \sum_{\alpha=0}^{v-1} d_\alpha P_\alpha\\
\text{imply} \quad c_\alpha = d_\alpha, \qquad \alpha = 0, 1, \ldots, v-1.
\end{gathered}
$$

Since Π is a group, $\Pi^{-1} = \Pi$. For any given i, let us put

$$\sum_{\Pi_i} P_\alpha = \sum_{\alpha=0}^{v-1} c_{i\alpha} P_\alpha \qquad \text{and} \qquad \sum_{\Pi_i} P_\alpha^{-1} = \sum_{\alpha=0}^{v-1} d_{i\alpha} P_\alpha .$$

Thus from condition (i) of (1.8) and (1.10), we get $c_{i\alpha} = d_{i\alpha}$, $\alpha = 0, 1,\ldots, v-1$; $i = 0, 1,\ldots, m$, and hence (i) of (1.9).

Let $p_{jk}(P_\alpha)$ be the number of solutions, (P_β, P_γ) with $P_\beta \in \Pi_j$ and $P_\gamma \in \Pi_k$, of the equation in (iii) of (1.9). Since Π is a group, $P_\beta P_\gamma$ is again in Π. Hence

$$\sum_{P_\beta \in \Pi_j} P_\beta \sum_{P_\gamma \in \Pi_k} P_\gamma = \sum_{\alpha=0}^{v-1} p_{jk}(P_\alpha)\, P_\alpha ,$$

and therefore, by condition (iii) of (1.8), we have

$$\sum_{\alpha=0}^{v-1} p_{jk}(P_\alpha)\, P_\alpha = \sum_{i=0}^{m} p_{jk}^{i} \sum_{\Pi_i} P_\alpha .$$

This implies, by (1.10), that

$$p_{jk}(P_\alpha) = p_{jk}^{i} .$$

for any P_α belonging to Π_i, $i = 0, 1,\ldots, m$. This shows that conditions (i), (ii), and (iii) of (1.8) are simply those of (1.9). Hence the theorem.

2. Association Relation Defined by a Group of Permutation Matrices of Order v

Let us number the rows and the columns of a permutation matrix of order v, by integers, $0, 1,\ldots, v-1$ and let these v integers correspond also to v objects on which we wish to define an m-class association scheme. Consider the group Π of permutation matrices of order v satisfying the conditions of (1.9). We shall say that *the object x is of the i-th relation to the object y* with respect to *the partition* $\{\Pi_i, i = 0, 1,\ldots, m\}$ of Π if there exists a $P_\alpha \in \Pi_i$ such that the element at the position (x, y) of P_α is 1. It follows that this relation is symmetric since $\Pi_i = \Pi_i^{-1}$, $i = 0, 1, 2,\ldots, m$. The matrix $A_i = \sum_{\Pi_i} P_\alpha$ of (1.5) gives us the pairs of objects (x, y) which are in the i-th relation, $i = 0, 1,\ldots, m$.

Let $G = \{g_0 = 1, g_1,\ldots, g_{v-1}\}$ be a finite group of order v. We shall write

$$\mathscr{g} = \begin{pmatrix} g_0 \\ g_1 \\ \cdot \\ g_{v-1} \end{pmatrix} \qquad \text{and} \qquad \mathscr{g}^\alpha = \mathscr{g}\, g_\alpha = \begin{pmatrix} g_0 g_\alpha \\ \vdots \\ g_{v-1} g_\alpha \end{pmatrix} = P_\alpha \mathscr{g}, \tag{2.1}$$

where g_α is an element of G and

$$P_\alpha = ((\delta_{x^\alpha, y})) \qquad x, y = 0, 1,\ldots, v-1 \tag{2.2}$$

is the $v \times v$ matrix corresponding to the linear transformation $g = g^{\alpha}$; $x^{\alpha} = z$ if and only if $g_x g_{\alpha} = g_z$; $\delta_{x,y} = 1$ if $x = y$ and $= 0$ if $x \neq y$, is the well-known Kronecker symbol.

Then $\Pi = \{P_{\alpha} ; \alpha = 0, 1,..., v-1\}$ is a regular group and G is isomorphic to Π ([3, 6]). Since Π is transitive and of order v, it follows that $\sum_{\alpha=0}^{v-1} P_{\alpha} = J_v$. Note that P_{α} is the permutation matrix corresponding to g_{α} and

$$\begin{aligned} &g_{\alpha} g_{\beta} = g_{\gamma} \rightleftarrows P_{\beta} P_{\alpha} = P_{\gamma} , \\ &g_0 = 1 \rightleftarrows P_0 = I,\ g_{\alpha}^{-1} \rightleftarrows P_{\alpha}^{-1} = P_{\alpha}'. \end{aligned} \tag{2.3}$$

An arbitrary set of elements of G is called a *complex* [3]. If A and B are two complexes in a group G, we write AB for the complex consisting of all elements ab, $a \in A$, $b \in B$ and call AB the product of A and B [3]. It is easy to verify the associative law for the multiplication of complexes [3].

Further A^{-1} will denote the complex $\{a^{-1} \mid a \in A\}$ where A is a complex of G. Let $f_g(A)$ denote the number of times the element g of G occurs in the collection (or complex) A of G. Then λA, where λ is a non-negative integer, is a collection H of G such that $f_g(H) = \lambda f_g(A)$ for every g in G. Again, if A and B are two collections (or complexes) of G, then $A + B$ is a collection K of G such that $f_g(K) = f_g(A) + f_g(B)$, for every g in G. Let

$$G = G_0 + G_1 + \cdots + G_m \tag{2.4}$$

be a partition of G into $(m + 1)$ complexes, where $G_0 = \{g_0 = 1\}$, 1 being the identity element of G. We then prove the following:

THEOREM 2.1. *A necessary and sufficient condition for the partition* (2.4) *of G to define an m-class association scheme on v elements is given by*

$$\begin{aligned} &\text{(i)}\quad G_i = G_i^{-1}, \qquad i = 0, 1,..., m, \\ &\text{(ii)}\quad G_j G_k = \sum_{i=0}^{m} p_{jk}^{i} G_i , \qquad j, k = 0, 1,..., m \end{aligned} \tag{2.5}$$

for some set of integers $\{p_{jk}^{i}\}$ $i, j, k = 0, 1,..., m$.

Proof. Consider the regular permutation group

$$\Pi = \{P_{\alpha} , \alpha = 0, 1,..., v - 1\}.$$

G is isomorphic to Π. From the correspondence mentioned in (2.3), it follows that the partition $\{G_i , i = 0, 1,..., m\}$ of G will correspond to the partition $\{\Pi_i , i = 0, 1,..., m\}$ of Π, where $P_{\alpha} \in \Pi_i$ if and only if $g_{\alpha} \in G_i$.

Hence the conditions (i) and (ii) of (2.5) are equivalent to (i) and (iii), respectively, of (1.9). Further, Π being a regular group of order v, $\sum_{\Pi} P_\alpha = J_v$, which is (ii) of (1.9). Thus it follows from Theorem 1.1 that conditions (i) and (ii) of (2.5) are necessary and sufficient for the partition (2.4) $\{G_i, i = 0, 1,..., m\}$ of G to define an m-class association scheme.

It may be remarked that the association relation between two objects (x, y) defined earlier through a permutation matrix can be defined equivalently in terms of the group elements as follows. Two elements x and y are in the i-th relation if and only if $g_x^{-1}g_y$ belongs to the complex G_i of the partition $\{G_i ; i = 0, 1, 2,..., m\}$ of G. The condition $G_i = G_i^{-1}$, $i = 0, 1,..., m$ implies that the relation is symmetric. It is not too difficult to verify that the integers P_{jk}^i of Theorem 2.1 will satisfy conditions (a) through (e) of (1.1).

3. Construction of Association Schemes from Finite Groups Examples

In this section, we construct several association schemes from different finite groups. In each case we give the group, its partition into complexes and the parameters, leaving the verification to the reader.

3.1. *Extended Group Divisible Association Scheme*

Let G be a finite group and suppose it is possible to partition it into complexes G_i's,

(3.1) $$G = G_0 + G_1 + \cdots + G_m ,$$

such that the complexes are defined by

(3.2) $$H_i = G_0 + G_1 + \cdots + G_i , H_m = G, i = 0, 1,..., m,$$

where H_i's form a chain of subgroups of G:

(3.3) $$H_0 \subset H_1 \subset \cdots \subset H_m = G.$$

Then it is easy to verify that the partition (3.1) satisfies the conditions of the Theorem 2.1 and hence gives an m-class association scheme.

Moreover, if we put $l_0 = 1$ and

(3.4) $$| G_i | = l_0 l_1 \cdots l_{i-1}(l_i - 1), i = 0, 1,..., m,$$

the parameters of the association scheme are given by

$$(3.5)\quad \begin{cases} p_{00}^{0} = 1,\ p_{0k}^{i} = \delta_{ik}, & i, k = 1,\dots, m, \\ p_{jk}^{i} = p_{kj}^{i} = \delta_{ik}(h_j - h_{j-1}) = \delta_{ik} l_0 l_1 \cdots l_{j-1}(l_j - 1), & \\ & j < k,\ j, k = 1,\dots, m, \\ p_{jj}^{i} = \begin{cases} h_j - h_{j-1} = l_0 l_1 \cdots l_{j-1}(l_j - 1), & 0 \leqslant i < j, \\ h_j - 2h_{j-1} = l_0 l_1 \cdots l_{j-1}(l_j - 2), & i = j, \\ 0, \text{ otherwise.} \end{cases} \end{cases}$$

The m-class association scheme thus constructed is the so-called extended group divisible association scheme.

In the case $m = 2$, this gives a 2-class association scheme of the group divisible type: Putting $h_0 = l_0 = 1$, $h_1 = l_1 = n$, $h_2 = mn$, $l_2 = m$, as in the usual notation, it follows from (3.5) that

$$(3.6)\quad \begin{cases} v = mn,\ n_1 = n - 1,\ n_2 = n(m - 1), \\ p_{11}^{1} = n - 2,\ p_{12}^{1} = p_{21}^{1} = 0,\ p_{22}^{1} = n(m - 1), \\ p_{11}^{2} = 0,\ p_{12}^{2} = p_{21}^{2} = n - 1,\ p_{22}^{2} = n(m - 2), \end{cases}$$

which gives the parameters of a 2-class association scheme of the group divisible type.

3.2. *Extended L_2 type Association Scheme*

Let H be a group of order n, and let G be the m-fold direct product of H:

$$(3.7)\quad G = H \times H \times \cdots \times H,$$

whose elements being denoted by $(a_1, \dots, a_m)$, $a_i \in H$, $i = 1,\dots, m$. It is then clear that

$$(3.8)\quad | G | = n^m,$$

and the unit element of G is $(1,\dots, 1)$, 1 being the unit element of H.

Now, let us put

$$(3.9)\quad \begin{cases} G_0 = \{(1,\dots, 1)\}, \\ G_1 = \{(a_1, 1,\dots, 1),\dots, (1,\dots, 1, a_m) \mid a_i \in H,\ a_i \neq 1\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ G_i = \text{the set of all elements of } G \text{ whose } i \text{ components are not the unit element and the rest } (m - i) \text{ components are the unit element,} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ G_m = \{(a_1,\dots, a_m) \mid a_i \in H,\ a_i \neq 1,\ i = 1,\dots, m\}. \end{cases}$$

It is obvious that

$$|G_i| = \binom{m}{i}(n-1)^i, \qquad i = 0, 1, \ldots, m, \tag{3.10}$$

and G is partitioned in the form

$$G = G_0 + G_1 + \cdots + G_m . \tag{3.11}$$

It is then not too difficult to check that the partition (3.11) satisfies the conditions of Theorem 2.1. The partition (3.11) thus gives us an m-class association scheme with parameters $v = n^m$ and p^i_{jk}, $i, j, k = 0, 1, \ldots m$ given by

$$p^i_{jk} = \sum_{u=\alpha(i,j,k)}^{\beta(i,j,k)} \frac{i!}{(i+j-k-2u)!\,u!(k-j+u)!} \times \binom{m-i}{k-i+u}(n-2)^{i+j-k-u}(n-1)^{k-i+u}, \tag{3.12}$$

where

$$p^i_{jk} = 0 \qquad \text{if} \quad \alpha(i,j,k) > \beta(i,j,k),$$

and

$$\begin{cases} \alpha(i,j,k) = \max(0, i-k), \\ \beta(i,j,k) = \min\left(i, m-k, \dfrac{i+j-k}{2}\right). \end{cases} \tag{3.13}$$

This gives us the values of p^i_{jk} for $j \leqslant k$. Since $p^i_{jk} = p^i_{kj}$, the above exhausts all the cases.

This may be regarded as an extension of the usual L_2 type association scheme. For $m = 2$, this gives us the 2-class association scheme of the L_2 type, with parameters

$$\begin{cases} p^0_{00} = p^0_{10} = p^0_{01} = p^0_{20} = p^0_{02} = p^0_{12} = p^0_{21} = 0, \\ p^0_{11} = 2(n-1),\ p^0_{22} = (n-1)^2,\ p^1_{10} = p^1_{01} = 1, \\ p^1_{02} = p^1_{20} = 0,\ p^1_{11} = (n-2),\ p^1_{12} = p^1_{21} = (n-1), \\ p^1_{22} = (n-1)(n-2),\ p^2_{00} = p^2_{01} = p^2_{10} = 0, \\ p^2_{20} = p^2_{02} = 1,\ p^2_{11} = 2,\ p^2_{12} = p^2_{21} = 2(n-2), \\ p^2_{22} = (n-2)^2. \end{cases} \tag{3.14}$$

3.3. *Extended T_2-type Association Scheme*

Let G be a finite group generated by the m generators

$$a_1, a_2, \ldots, a_m, \tag{3.15}$$

subject to the generating relations

$$a_i^2 = 1 \text{ and } a_i a_j, \; i, j = 1, \ldots, m. \tag{3.16}$$

Let us define

$$\begin{cases} G_0 = 1, \\ G_1 = a_1, \ldots, a_m, \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ G_i = \{a_{j_1} a_{j_2} \cdots a_{j_i} \mid 1 \leqslant j_1 < \cdots < j_i \leqslant m\}, \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ G_m = \{a_1 \cdots a_m\}. \end{cases} \tag{3.17}$$

Then, these subsets of G form a partition of G:

$$G = G_0 + G_1 + \cdots + G_m, \tag{3.18}$$

for which it is evident that

$$| G | = 2^m, \; | G_i | = \binom{m}{i}, \; i = 0, 1, \ldots, m. \tag{3.19}$$

The group thus defined is isomorphic to the additive group V_m, consisting of all binary m-vectors whose components are from the Galois field GF (2).

It is easy to verify that the partition (3.18) satisfies the conditions of Theorem 2.1 and hence gives us an m-class association scheme with parameters $v = 2^m$ and p^i_{jk} given by

$$p^i_{jk} = \binom{i}{\frac{i+j-k}{2}} \binom{m-i}{\frac{j+k-i}{2}}, \text{ if } i+k-j \geqslant 0, \text{ and both } \frac{i+j-k}{2} \text{ and } \frac{j+k-i}{2} \text{ are non-negative integers, } 0, \text{ otherwise.} \tag{3.20}$$

It is easy to see that

$$\bar{G} = G_0 + G_2 + G_4 + \cdots + G_{2i} + \cdots + G_{2[m/2]}, \tag{3.21}$$

[ ] being the usual Gauss symbol, is a subgroup of order 2^{m-1} and

$$(3.22)\quad \begin{cases} G_{2j}^2 = \sum_{i=0}^{s} \binom{2i}{i}\binom{m-2i}{2j-i} G_{2i}, \qquad j = 0, 1,\ldots, s, \\ G_{2j}G_{2k} = \sum_{i=\max(0,j-k)}^{j+k} \binom{2i}{i+j-k}\binom{m-2i}{j+k-i} G_{2i}, \\ \qquad\qquad\qquad\qquad j, k = 0, 1,\ldots, s, \end{cases}$$

where $s = [m/2]$.

The partition (3.21) gives us an s-class association scheme with parameters given by (3.22).

From (3.22) we see that

$$(3.23)\qquad G_2{}^2 = \binom{n}{2} G_0 + 2(n-2) G_2 + 6G_4 .$$

We shall say that two elements x, y of G_2 are 1st associates or 2nd associates according, respectively, as $x^{-1} y \in G_2$ or as $x^{-1} y \in G_4$. Then, it is easy to check that the resulting association scheme is of the usual triangular type, whose parameters are

$$(3.24)\quad \begin{cases} v = \binom{n}{2}, n_1 = 2(n-2), n_2 = \binom{n-2}{2}, \\ p_{11}^1 = n-2, p_{12}^1 = p_{21}^1 = n-3, p_{22}^1 = \binom{n-3}{2}, \\ p_{11}^2 = 4, p_{12}^2 = p_{21}^2 = 2(n-4), p_{22}^2 = \binom{n-4}{2}. \end{cases}$$

3.4. *Balanced Incomplete Block Design and Association Scheme from an Abelian Group*

Let G be an Abelian group of order V, and suppose that partition

$$(3.25)\qquad G = G_0 + G_1 + \cdots + G_m ,$$

where $G_0 = \{1\}$, 1 being the unit element of G, satisfies the following conditions:

$$(3.26)\quad \begin{cases} \text{(i)} \quad |G_i| = n, \qquad i = 1,\ldots, m, \\ \text{(ii)} \quad G_0 + G_i \text{ is a subgroup of } G, \qquad i = 1,\ldots, m, \qquad \text{and} \\ \text{(iii)} \quad \text{for any given } i \text{ and } j\ (i \neq j), \text{ there exists } n \text{ integers} \\ \qquad k(i,j)_u, \qquad u = 1,\ldots, n, \qquad \text{such that} \end{cases}$$

$$G_iG_j = \sum_{u=1}^{n} G_{k(i,j)_u} .$$

Then, it is clear that $v = nm + 1$, and $(n + 1)^2$ divides v since $(G_0 + G_i)(G_0 + G_j)$ is a subgroup of G.

We first note that a sufficient condition for (iii) in (3.26) to be satisfied under the conditions (i) and (ii) is that each $G_0 + G_i$ has no proper subgroup other than G_0.

It is also seen easily that under the conditions given by (3.26), the complex

$$H(i, j) = G_0 + G_i + G_j + \sum_{u=1}^{n} G_{k(i,j)_u} \tag{3.27}$$

forms a subgroup of G, for $i \neq j$. It is also a minimal subgroup of G, which contains $G_i + G_j$. We state three lemmas (without proof):

LEMMA 3.1. *Suppose, for the partition* (3.25) *of* G, *conditions* (i) *and* (ii) *of* (3.26) *are satisfied. Then, for condition* (iii) *of* (3.26) *it is necessary and sufficient that, for any given* i *and* j, $i \neq j$, *there exist a set of* n *integers* $k(i, j)_u$, $u = 1,..., n$, *such that the subset* $H(i, j)$ *defined by* (3.27) *forms a subgroup of* G.

LEMMA 3.2. *Under the same situation as in the above lemma, suppose that, for a set of* $(n + 2) G_i$*'s* $\{G_{i_j}\}$, $j = 1,..., n + 2$,

$$G_0 + G_{i_1} + \cdots + G_{i_{n+2}} \tag{3.28}$$

forms a subgroup of G. *Then, for any given* j_1 *and* j_2 *distinct, it holds that*

$$G_{i_{j_1}} G_{i_{j_2}} = \sum_{j \neq j_1, j_2} G_{i_j}. \tag{3.29}$$

It is now evident that, under the conditions of (3.26) for any given i and j distinct, there exists a unique set of n G_i's satisfying the equality given in (iii) of (3.26). Simplifying the notation of suffix, let them be

$$\mathscr{H}(i, j) = \{G_i, G_j, G_{k_1}, ..., G_{k_n}\}. \tag{3.30}$$

Lemmas 3.1 and 3.2 say, then, that this set of G_i's is determined uniquely by giving any pair of G_i's in the set. The number of distinct $H(i, j)$'s under (3.25) and (3.26) is then given by $m(m - 1)/(n + 2)(n + 1)$ and this must be an integer. Further, it is easy to verify that any pair of $H(i, j)$'s in the family of all mutually distinct $H(i, j)$'s do not possess more than one G_k in common.

Under (3.25) and (3.26), $(n + 1)^2$ divides $v = nm + 1$. This implies that

$$m = t + (tn + 1)(n + 2), \qquad t \text{ an integer}. \tag{3.31}$$

Then, $(m-1)/(n+1) = 1 + t(n+1)$ is an integer.

Further, both n and $n+1$ divide $n(m-1)$. Hence, $n(m-1)$ is an even integer and $n(m-1)/2$ is a positive integer.

A given G_k is contained in exactly $(m-1)/(n+1)$ distinct $H(i,j)$'s. This gives us

LEMMA 3.3. *Under the conditions of* (3.25) *and* (3.26),

$$\sum_{i<j} G_i G_j = \frac{n(m-1)}{2} \sum_{i=1}^{m} G_i . \tag{3.32}$$

Thus we have seen that each G_k is contained in exactly $(m-1)/(n+1)$ distinct $\mathscr{H}(i,j)$'s. It is also clear that any given pair G_k and G_u are contained in exactly one $\mathscr{H}(i,j)$ belonging to $\mathscr{B}$, the family of all distinct $\mathscr{H}(i,j)$'s.

We are now in a position to state the following:

THEOREM 3.1. *Suppose that conditions* (i) *through* (iii) *of* (3.26) *are satisfied for a partition* (3.25) *of an Abelian group G of order $nm+1$. Then, the family $\mathscr{B}$ is a "(v, k, λ)-configuration"* [4], *where*

$$v = m,\ k = n+2,\ \lambda = 1, \tag{3.33}$$

and the other parameters are given by

$$b = \frac{m(m-1)}{(n+1)(n+2)},\ r = \frac{m-1}{n+1}. \tag{3.34}$$

It should be remarked that the configuration of this theorem is a balanced incomplete block design with the parameters given above. Such a design can be actually constructed if there exists an Abelian group for which conditions (3.25) and (3.26) are satisfied.

Note that, for $t = 1$ in (3.31), we have

$$v = m = 1 + (n+1)(n+2) = b,\ r = k = n+2,\ \lambda = 1, \tag{3.35}$$

which correspond to a PG(2, $n+1$) if one exists. For $t \neq 1$, for b to be an integer, $n+2$ must divide $t(t-1)$. We do not have an example of a design in this category.

THEOREM 3.2. *If there exists an Abelian group G with the partition*

(3.25) *satisfying the conditions of* (3.26), *then there can be found an m-class association scheme, whose parameters are given by*

$$v = nm + 1, n_i = n, i = 1,..., m,$$

$$(3.36) \qquad p^i_{jk} = \begin{cases} 1, & \text{if } G_i, G_j \text{ and } G_k \text{ belongs to the same class in } \mathscr{B}, \\ & \text{and } i \neq j, i \neq k, j \neq k \text{ (i.e., } G_i, G_j \text{ and } G_k \text{ are distinct)}, \\ n-1, & \text{if } i = j = k, \\ 0, & \text{otherwise}, \quad (i, j, k = 1,..., m). \end{cases}$$

Proof. By the preceding theorem, we can see that, for any given j and k distinct, there exists exactly one class, $\mathscr{H}(t, u)$ say, in $\mathscr{B}$ such that $\mathscr{H}(j, k) = \mathscr{H}(t, u)$.

Hence the theorem follows from Lemma 3.2.

In the case $n = s - 1$ and $v = s^t$, where s is any prime power and t is any positive integer greater than 2, and hence $m = (s^t - 1)/(s - 1)$, the association scheme given in the above theorem is the *geometric association scheme* given by Y. Fujii [2], basing upon the structure of finite projective geometry.

REFERENCES

1. I. M. CHAKRAVARTI AND W. C. BLACKWELDER, On some composition and extension methods in the construction of block designs from association matrices, "Combinatorial Mathematics and Its Applications" (Proceedings) (R. C. Bose and T. A. Dowling, eds.), Univ. of North Carolina Press, Chapel Hill, 1969.
2. Y. FUJII, Geometrical association schemes and fractional factorial designs, *J. Sci. Hiroshima Univ. Ser. A-I Math.* **31** (1967), 195–209.
3. M. HALL, "The Theory of Groups," Macmilan, New York, 1959.
4. H. J. RYSER, "Combinatorial Mathematics" (Carus Monograph 14), Wiley, New York, 1963.
5. W. A. THOMPSON, JR. A note on PBIB design matrices, *Ann. Math. Statist.* **29** (1958), 919–922.
6. H. WIELANDT, "Finite Permutation Groups," Academic Press, New York, 1964.